OCR RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT

4726/01

MATHEMATICS

Further Pure Mathematics 2

TUESDAY 16 JANUARY 2007

Morning

Time: 1 hour 30 minutes

Additional Materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

1 It is given that $f(x) = \ln(3 + x)$.

(i) Find the exact values of $f(0)$ and $f'(0)$, and show that $f''(0) = -\frac{1}{9}$. [3]

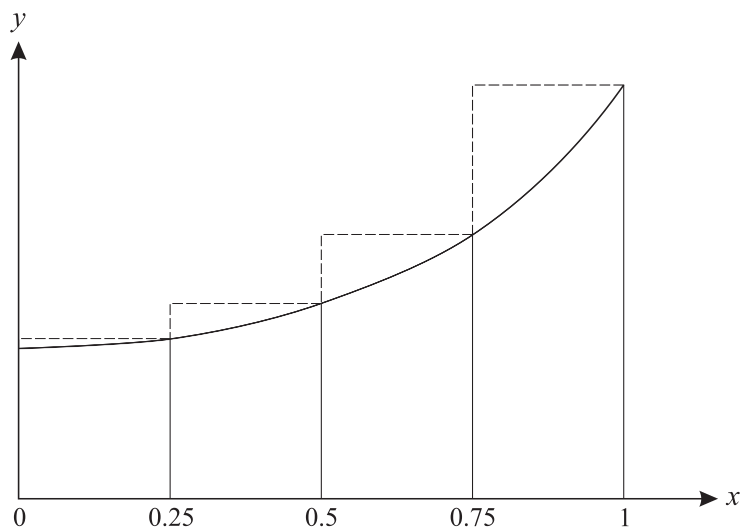
(ii) Hence write down the first three terms of the Maclaurin series for $f(x)$, given that $-3 < x \leqslant 3$. [2]

2 It is given that $f(x) = x^2 - \tan^{-1} x$.

(i) Show by calculation that the equation $f(x) = 0$ has a root in the interval $0.8 < x < 0.9$. [2]

(ii) Use the Newton-Raphson method, with a first approximation 0.8, to find the next approximation to this root. Give your answer correct to 3 decimal places. [4]

3



The diagram shows the curve with equation $y = e^{x^2}$, for $0 \leqslant x \leqslant 1$. The region under the curve between these limits is divided into four strips of equal width. The area of this region under the curve is A.

(i) By considering the set of rectangles indicated in the diagram, show that an upper bound for A is 1.71. [3]

(ii) By considering an appropriate set of four rectangles, find a lower bound for A. [3]

4 **(i)** On separate diagrams, sketch the graphs of $y = \sinh x$ and $y = \operatorname{cosech} x$. [3]

(ii) Show that $\operatorname{cosech} x = \dfrac{2e^x}{e^{2x} - 1}$, and hence, using the substitution $u = e^x$, find $\int \operatorname{cosech} x \, dx$. [6]

5 It is given that, for non-negative integers n,

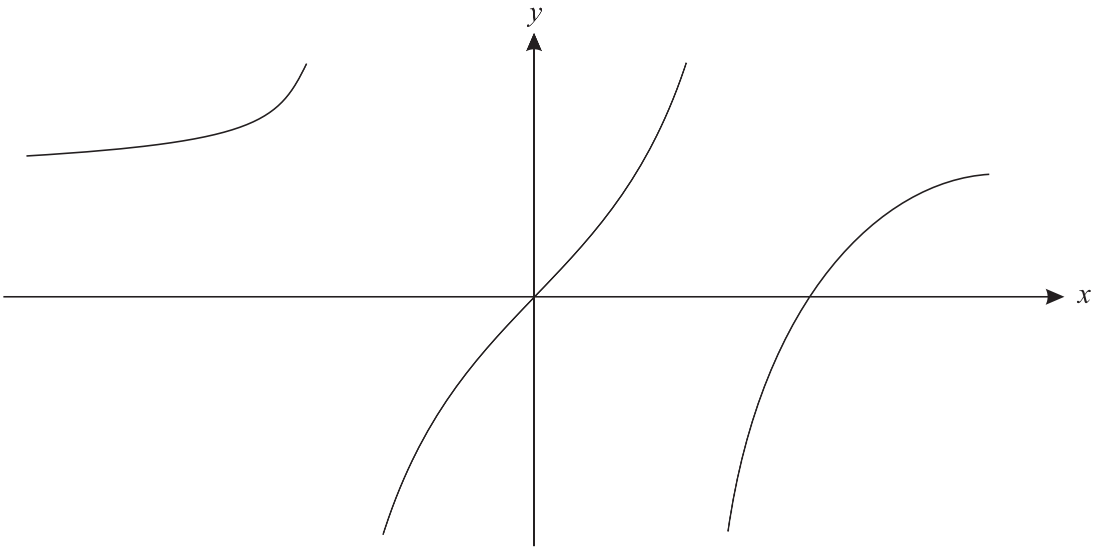
$$I_n = \int_0^{\frac{1}{2}\pi} x^n \cos x \, dx.$$

(i) Prove that, for $n \geqslant 2$,

$$I_n = \left(\tfrac{1}{2}\pi\right)^n - n(n-1)I_{n-2}.$$ [5]

(ii) Find I_4 in terms of π. [4]

6



The diagram shows the curve with equation $y = \frac{2x^2 - 3ax}{x^2 - a^2}$, where a is a positive constant.

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Sketch the curve with equation

$$y^2 = \frac{2x^2 - 3ax}{x^2 - a^2}.$$

State the coordinates of any points where the curve crosses the axes, and give the equations of any asymptotes. [5]

7 **(i)** Express $\frac{1 - t^2}{t^2(1 + t^2)}$ in partial fractions. [4]

(ii) Use the substitution $t = \tan \frac{1}{2}x$ to show that

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{\cos x}{1 - \cos x} \, dx = \sqrt{3} - 1 - \tfrac{1}{6}\pi.$$ [5]

8 **(i)** Define $\tanh y$ in terms of e^y and e^{-y}. [1]

(ii) Given that $y = \tanh^{-1} x$, where $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. [3]

(iii) Find the exact solution of the equation $3\cosh x = 4\sinh x$, giving the answer in terms of a logarithm. [2]

(iv) Solve the equation

$$\tanh^{-1} x + \ln(1-x) = \ln\left(\tfrac{4}{5}\right).$$ [3]

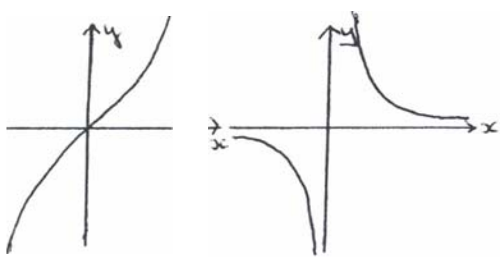
9 The equation of a curve, in polar coordinates, is

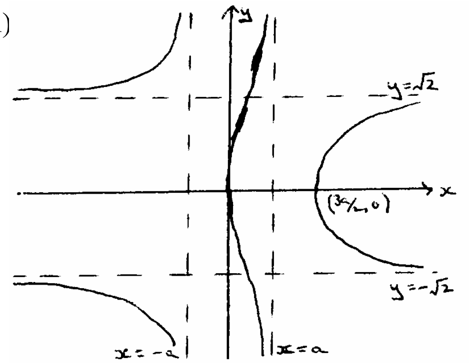
$$r = \sec\theta + \tan\theta, \quad \text{for } 0 \leqslant \theta \leqslant \tfrac{1}{3}\pi.$$

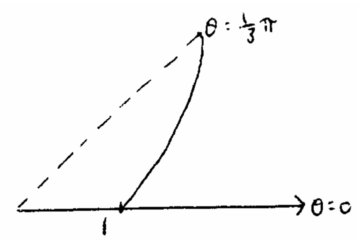
(i) Sketch the curve. [2]

(ii) Find the exact area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [6]

(iii) Find a cartesian equation of the curve. [3]

1 (i) f(O) = In 3 f	Bl
f'(0) = $^1/_3$	Bl
f"(O) = - $^1/_9$ *A.G.*	B1 Clearly derived
(ii) Reasonable attempt at Maclaurin	Ml Form In3 + $ax + bx^2$, with a,b related to f " f"
f(x) = ln 3 + $^1/_3x$-$^1/_{18}x^2$	*Al* √*J* On their values off' and f"
	SR Use *ln(3+x)* = In3 + In(1 +$^1/_3$ *x)* Ml Use Formulae Book to get
	In3 + *Y3X - Y2(VJX)2* =
	In3 + *Y3X - 1/lgX2* Al
2 (i) f(0.8) = - 0.03, f(0.9) = +0.077 (accurately e.g. accept -0.02 t0 -0.04)	B1
Explain (change of sign, graph etc.)	B1
	SR Use x = √*J(tan^{-l}x)* and compare x to √J(tan^{-l} x) for x=*0.8,* 0.9 B 1
	Explain "change in sign" B 1
(ii) Differentiate two terms	B1 Get *2x* - I *I(* 1 *+x^2)*
Use correct form of Newton-Ra ph son with 0.8, using their f *'(x)*	Ml 0.8 - *f(0.8)/f '(0.8)*
Use their N-R to give one more approximation to 3 d.p. minimum	Ml√
Get x = 0.835	Al 3d.p. - accept answer which rounds
3 (i) Show area of rect. = $^1/_4$(e$^{1/16}$ + e$^{1/4}$ + e$^{9/16}$ + e^1)	Ml Or numeric equivalent
Show area = 1.7054	Al At least 3 d.p. correct
Explain the < 1.71 in terms of areas	Bl AG. Inequality required
(ii) Identify areas for > sign	B1 Inequality or diagram required
Show area of rect. = $^1/_4$ *(e^o + e^{lll6}* + e$^{1/4}$ + e$^{9/16}$)	Ml Or numeric evidence
Get A > 1.27	Al cao; or answer which rounds down
4 (i) 	BI Correct shape for sinh x
	B1 Correct shape for cosech x
	B1 Obvious point *(dy/dx* ≠O)/asymptotes clear
(ii) Correct definition of sinh x	B1 May be implied
Invert and mult. by *eX* to AG.	B1 Must be clear; allow *2/(eX* - e *-X)* as mimimum simplification
Sub. $u = e^x$ and $du = e^x dx$	M1 Or equivalent, all x eliminated and not $dx = du$
Replace to *2/(u^2* - 1) *du*	Al
Integrate to *aln((u - l)/(u* + 1)	A1√ Use formulae book, PT, or atanh^{-1}u
Replace u	Al No need for c

5 (i) Reasonable attempt at parts Get	M1 Involving second integral Al
xnsin x - ∫ sin x. nx^{n-1} dx	M1
Attempt parts again Accurately	Al
Clearly derive AG.	A1 Indicate $(^1/_2\pi)^n$ and 0 from limits
(ii) Get $I_4 = (^1/_2\pi)^4 - 12I_2$ or $I_2 = (^1/_2\pi)^2 - 2I_0$	B1
Show clearly $I_0 = 1$	B1 May use I_2
Replace their values in relation Get	M1
$I_4 = {^1/_{16}}\pi^4 - 3\pi^2 + 24$	A1 cao
6 (i) $x = \pm a$, $y = 2$	B1, B1, B1 Must be =; no working needed
(ii) 	B1 Two correct labelled asymptotes ∥ Ox and approaches B1 Two correct labelled asymptotes ∥ Oy and approaches B1 Crosses at $(^3/_2a, 0)$ (and (0,0) - may be implied B1 90° where it crosses Ox; smoothly B1 Symmetry in Ox
7 (i) Write as $A/t + B/t^2 + (Ct + D)/(t^2 + 1)$	M1 Allow $(At+B)/t^2$; justify $B/t^2 + D/(l + t^2)$ if only used
Equate $At(t^2+1) + B(t^2+1) + (Ct+D)t^2$ to $1 - t^2$	M1√
Insert t values / equate coeff.	M1 Lead to at least two constant values
Get A = C = 0, B = L D =-2	Al
	SR Other methods leading to correct PF can earn 4 marks; 2 M marks for reasonable method going wrong
(ii) Derive or quote cos x in terms of t	Bl
Derive or quote dx = 2 $dt/(1 + t^2)$	B1
Sub. in to correct P.F.	M1 Allow $k\ (l-t^2)/((t^2(l + t^2))$ or equivalent
Integrate to $-1/t - 2\tan^{-1}t$	Al√ From their k
Use limits to clearly get AG.	Al
8 (i) Get $(e^y - e^{-y})/(e^y + e^{-y})$	B1 Allow $(e^{2Y}-1)/(e^{2y}+ 1)$ or if x used
(ii) Attempt quad. in e^Y	M1 Multiply by e^Y and tidy
Solve for e^Y	M1
Clearly get AG.	Al
(iii) Rewrite as tanh $x = k$	M1 SR Use hyp def^n to get quad. in e^X M I
Use (ii) for x = ½ 1n 7 or equivalent	Al Solve e^{2x} = 7 for x to $^1/_2$ 1n 7 Al
(iv) Use of log laws	Bl One used correctly
Correctly equate 1n A = 1n B to $A = B$	M1 Or $1n(^A l_B) = 0$
Get $x = \pm\ ^3/_5$	Al

9 (i)		B1 Shape for correct θ; ignore other θ Used; start at $(r,0)$ B1 θ=0, r=1 and increasing r
(ii)	Use correct formula with correct r	B1
	$\int \sec^2 x\ dx = \tan x$ used	B1
	Quote $\int 2 \sec x \tan x\ dx = 2\sec x$	B1 Or sub. correctly
	Replace $\tan^2 x$ by $\sec^2 x - 1$ to integrate	M1
	Reasonable attempt to integrate 3 terms And to use limits correctly	M1
	Get $\sqrt{3} + 1 - \frac{1}{6}\pi$	A1 Exact only
(iii)	Use $x = r\cos\theta$, $y = r\sin\theta$, $r = (x^2+y^2)^{1/2}$	M1
	Reasonable attempt to eliminate r,θ	M1
	Get $y = (x-1)\sqrt{(x^2+y^2)}$	A1 Or equivalent